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ON THE ASYMPTOTIC BEHAVIOR OF AN UNLOADING WAVE

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The problem of unloading wave propagation in a cylindrical, semi-infinite, elastic-plastic rod is considered [1] and the asymptotic properties of this problem are studied for large values of time and large distances from the end of the rod. It is assumed that the curve exhibiting the relation between the stress and strain under loading has an initial linear section while the nonlinear part of this curve is turned convexley to the stress axis. The unloading occurs along lines parallel to the initial linear section of the loading curve. By the condition of the problem under consideration, the normal stress at the end of the rod grows from zero to the maximum value exceeding the proportionality limit σ_s during a certain time or instantaneously, and then decreases monotonically to some value $p_e \geqslant 0$ in a finite or infinite time. The cases $p_e > \sigma_s$, $p_e = \sigma_s$, and $p_e < \sigma_s$ are examined.

1. Let t be the time, h the Lagrange coordinate representing the distance between the rod cross section and the base normal to the rod lateral surface at the initial time $t = 0$, u the velocity, σ the stress (technical), ε the strain, and ρ_0 the initial density of the rod material.

The dependence $\sigma(\varepsilon)$ according to which loading is realized, is linear in the interval $0 \leqslant \varepsilon \leqslant \varepsilon_s$ and nonlinear for $\varepsilon > \varepsilon_s$. For $0 \leqslant \varepsilon \leqslant \varepsilon_s$ we have $\sigma'(\varepsilon) = E$ ($E = \text{const}$ is the elastic modulus), while for $\varepsilon > \varepsilon_s$ the function $\sigma(\varepsilon)$ possesses the properties $0 < \sigma'(\varepsilon) < E$, $-\infty < \sigma''(\varepsilon) < 0$. The right and left derivatives of the function $\sigma(\varepsilon)$ are equal at the point $\varepsilon = \varepsilon_s$ if the order of the derivatives is $n - 1$ or lower, and differ if the order of the derivatives is n, $n \geqslant 2$, $-\infty < \sigma^{(n)}(\varepsilon_s + 0) < 0$. Unloading occurs according to the linear law $\sigma - \sigma_* = E(\varepsilon - \varepsilon_*)$, where $\sigma_* = \sigma_*(h)$, $\varepsilon_* = \varepsilon_*(h)$ are the maximal stress and strain for the element noted by the coordinate h.

The normal stress at the end of the rod $h = 0$ varies according to the known law $\sigma = p(t)$. The function $p(t)$ grows monotonically from zero to the maximum value $\sigma_m > \sigma_s$ ($\sigma_s = E\varepsilon_s$) in the interval $0 \leqslant t \leqslant \tau$, and then decreases monotonically, where it either tends to a certain value $p_e \geqslant 0$ as $t \to \infty$, or becomes equal to $p_e \geqslant 0$ for a certain finite value of the argument, after which $p(t) \equiv p_e$. The normal stress at the end of the rod can grow from zero to σ_m instantaneously ($\tau = 0$).

We have the rest domain O, the loading domain 1, and the unloading domain 2 in the ht plane (Fig. 1). The line $h = g_0 t$ ($g_0 = \sqrt{E/\rho_0}$) is the boundary between the rest and loading domains, while the unloading line $h = \varphi(t)$ is the boundary between the loading and unloading domains, which emerges from the point with the coordinates $h = 0,\ t = \tau$.

The simple wave

$$h = g\,[t - F(\sigma)], \quad u = -\frac{1}{\rho_0}\int_0^{\sigma}\frac{d\sigma}{g}, \quad g = \sqrt{\frac{\sigma'(\varepsilon)}{\rho_0}} \tag{1.1}$$

where $F(\sigma)$ is the inverse function, in the interval $0 \leqslant \sigma \leqslant \sigma_m$, to the function $p(t)$ considered in the interval $0 \leqslant t \leqslant \tau$, is the solution in the loading domain. If $\tau = 0$, then $F \equiv 0$. In the unloading domain we have

$$-u = f_1(\alpha) + f_2(\beta) \tag{1.2}$$

$$\sigma / \rho_0 g_0 = f_1(\alpha) - f_2(\beta)$$

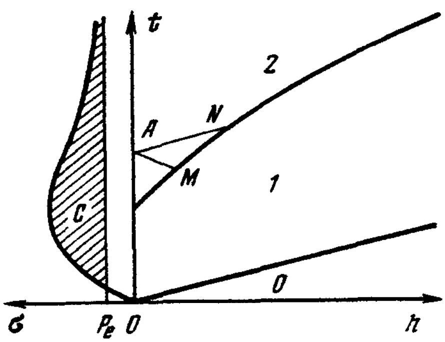

Fig. 1

where $f_1(\alpha)$ and $f_2(\beta)$ are functions of the charactersitic variables $\alpha = g_0 t - h$ and $\beta = g_0 t + h$.

On the unloading line α, g and the other quantities can be considered functions of the variable t. Let $\alpha^*(t)$, $g^*(t)$, etc., or for brevity, simply α^*, g^*, ... denote these dependences.

Using the known inequalities $\varphi'(t) \geqslant g^*(t)$, $\varphi'(t) \leqslant g_0$ [2], we arrive at the deduction that $\varphi(t)$, $\beta^*(t)$ and $\alpha^*(t)$ are monotonically increasing functions, where $\varphi(t)$ and $\beta^*(t)$ are strictly monotonically increasing functions. As $t \to \infty$ we have $\varphi(t) \to \infty$ and $\beta^*(t) \to \infty$. Since $\beta^*(t)$ is a strictly monotonic function, then its strictly monotonic inverse function $t = \chi(\beta)$ exists.

Differentiating the following equation with respect to the variable t

$$\varphi(t) = g^*\,[t - F(\sigma^*)] \tag{1.3}$$

and then using the inequality $\varphi'(t) \geqslant g^*(t)$, we see that $d\sigma^* / dt \leqslant 0$, $d\varepsilon^* / dt \leqslant 0$, $dg^* / dt \geqslant 0$. Therefore, the functions $\sigma^*(t)$ and $\varepsilon^*(t)$ are monotonically decreasing while $g^*(t)$ is a monotonically increasing function. The limits of the functions $\sigma^*(t)$, $\varepsilon^*(t)$ and $g^*(t)$ as $t \to \infty$ will be denoted by σ_e, ε_e, g_e, respectively. It is shown in [3, 4] that for any t from the band $\tau \leqslant t < \infty$ the inequality $\sigma^*(t) > \sigma_s$ is satisfied. Hence $\sigma_e \geqslant \sigma_s$. If $\sigma_e = \sigma_s$, then $\sigma^*(t) > \sigma_e$ for any t from the band $\tau \leqslant t < \infty$. It will be shown in Sect. 2 that the function $\sigma^*(t)$ possesses such a property even for $\sigma_e > \sigma_s$.

Let us introduce the functions

$$\psi(\beta) = \sigma^*[\chi(\beta)], \quad \Lambda(\beta) = \psi(\beta) - \sigma_e$$

It is clear that $\psi(\beta^*) \equiv \sigma^*(t)$, $\Lambda(\beta^*) \equiv \sigma^*(t) - \sigma_e$. As $\beta \to \infty$ we have $\psi(\beta) \to \sigma_e$, $\Lambda(\beta) \to 0$.

From the continuity condition for the stress and velocity on the unloading line we obtain

$$f_1(\alpha^*) = \frac{1}{2\rho_0 g_0}\int_0^{\sigma^*}\frac{g_0 + g}{g}\,d\sigma, \quad f_2(\beta^*) = \frac{1}{2\rho_0 g_0}\int_0^{\sigma^*}\frac{g_0 - g}{g}\,d\sigma$$

It can therefore be written that

$$f_1(\alpha^*) = \frac{1}{2\rho_0 g_0}\left[\int_0^{\sigma_e} \frac{g_0+g}{g}\,d\sigma + \frac{g_0+g_e}{g_e}\Lambda(\beta^*) + o(\Lambda(\beta^*))\right], \quad t\to\infty \quad (1.4)$$

$$f_2(\beta^*) = \frac{1}{2\rho_0 g_0}\left[\int_0^{\sigma_e} \frac{g_0-g}{g}\,d\sigma + \frac{g_0-g_e}{g_e}\Lambda(\beta^*) + o(\Lambda(\beta^*))\right], \quad t\to\infty \quad (1.5)$$

The representation which we shall use for $\sigma_e > \sigma_s$

$$f_2(\beta) = \frac{1}{2\rho_0 g_0}\left[\int_0^{\sigma_e} \frac{g_0-g}{g}\,d\sigma + \frac{g_0-g_e}{g_e}\Lambda(\beta) + \theta_1(\beta)\Lambda(\beta)\right] \quad (1.6)$$

$$\theta_1(\beta)\to 0, \quad \text{for} \quad \beta\to\infty$$

follows from the last formula.

When $\sigma_e = \sigma_s$, we analogously obtain the representation

$$f_2(\beta) = \frac{1}{2\rho_0 g_0}[k_n\Lambda^n(\beta) + \theta_n(\beta)\Lambda^n(\beta)] \quad (1.7)$$

$$\theta_n(\beta)\to 0 \;\text{ for }\; \beta\to\infty, \quad k_n = -\frac{g^{(n-1)}(\sigma_s+0)}{n!\,g_0} = -\frac{\sigma^{(n)}(e_s+0)}{n!\,2\rho_0^{\,n} g_0^{\,2n}}, \quad k_n>0$$

2. The main results of this investigation result from the momentum equation written in the integral form

$$\int_0^t p(t)\,dt = I_1(t) + I_2(t) \quad (2.1)$$

$$I_1(t) = -\rho_0\int_{\varphi(t)}^{g_e t} u\,dh, \quad I_2(t) = -\rho_0\int_0^{\varphi(t)} u\,dh$$

We convert the terms $I_1(t)$ and $I_2(t)$ in the right side of this equation.

Let us consider the first term. Because of integration by parts and subsequent application of (1.1) and (1.3), we obtain

$$I_1(t) = \sigma_e t - \varphi(t)\int_0^{\sigma_e}\frac{d\sigma}{g} - \int_0^{\sigma^*} F(\sigma)\,d\sigma + S_1(t) + S_2(t)$$

$$S_1(t) = t\left[(\sigma^*-\sigma_e) - g^*\int_{\sigma_e}^{\sigma^*}\frac{d\sigma}{g}\right], \quad S_2(t) = g^* F(\sigma^*)\int_{\sigma_e}^{\sigma^*}\frac{d\sigma}{g}$$

where

$$S_1(t) \sim (n-1) k_n t \Lambda^n (\beta^*), \quad S_2(t) \sim F(\sigma_s)\Lambda(\beta^*), \quad t \to \infty \tag{2.2}$$

$(\sigma_e = \sigma_s)$

$$S_1(t) \sim l_2 t \Lambda^2 (\beta^*), \quad S_2(t) \sim F(\sigma_e)\Lambda(\beta^*), \quad t \to \infty \tag{2.3}$$

$$l_2 = -\frac{g'(\sigma_e)}{2g_e} = -\frac{\sigma''(e_e)}{4\rho_0^2 g_e^4} \qquad (\sigma_e > \sigma_s)$$

We turn to the second term. On the basis of (1.2) and the boundary condition at the end of the rod we represent the integrand as follows:

$$u = -\frac{1}{\rho_0 g_0} p\left(\frac{\alpha}{g_0}\right) - f_2(\alpha) - f_2(\beta)$$

After the conversion we obtain

$$I_2(t) = \rho_0 \int_{\alpha^*}^{\beta^*} f_2(\beta)\, d\beta + \int_{\alpha^*/g_0}^{t} p(t)\, dt$$

Let us introduce the function

$$\omega(t) = \int_0^t (p - p_e)\, dt + \int_0^{p_e} F(\sigma)\, d\sigma$$

for which the estimate $\omega(t) = o(t)$, $t \to \infty$ is evident.

Applying this function and taking account of the expressions obtained for $I_1(t)$ and $I_2(t)$, we represent (2.1) in the form

$$\omega\left(\frac{\alpha^*}{g_0}\right) = (\sigma_e - p_e)\frac{\alpha^*}{g_0} + \varphi(t) \int_{\sigma_s}^{\sigma_e} \left(\frac{1}{g_0} - \frac{1}{g}\right) d\sigma + \tag{2.4}$$

$$\rho_0 \int_{\alpha^*}^{\beta^*} f_2(\beta)\, d\beta + S_1(t) + S_2(t) + \Delta, \quad \Delta = -\int_{p_e}^{\sigma^*} F(\sigma)\, d\sigma$$

Furthermore, if the representation (1.6) is used, we will have

$$\omega(\alpha^*/g_0) = (\sigma_e - p_e)\alpha^*/g_0 + R_1(t) + R_2(t) + S_1(t) + S_2(t) + \Delta$$

$$R_1(t) = \frac{g_0 - g_e}{2g_0 g_e} \int_{\alpha^*}^{\beta^*} \Lambda(\beta)\, d\beta, \quad R_2(t) = \frac{1}{2g_0} \int_{\alpha^*}^{\beta^*} \theta_1(\beta)\Lambda(\beta)\, d\beta$$

For $g_0 \tau \leqslant \beta < \infty$ the function $\Lambda(\beta)$ is positive and monotonically decreasing. Hence, the inequality

$$\int_{\alpha^*}^{\beta^*} \Lambda(\beta)\, d\beta \geqslant 2\Lambda(\beta^*)\varphi(t)$$

can be written, whereupon we have the estimates $S_1(t) = o(R_1(t))$, $S_2(t) = o(R_1(t))$, $t \to \infty$. Taking these estimates into account, we obtain the following equation for the case $\sigma_e > \sigma_s$

$$\omega(\alpha^*/g_0) = (\sigma_e - p_e)\,\alpha^*/g_0 + [1 + r(t)]\,R_1(t) + R_2(t) + \Delta \qquad (2.5)$$
$$r(t) \to 0 \quad \text{for} \quad t \to \infty$$

We hence see here that by applying the momentum equation in integral form the function $\sigma^*(t)$ possesses the following property mentioned in Sect. 1 for not only $\sigma_e = \sigma_s$ but also for $\sigma_e > \sigma_s$: For any t from the range $\tau \leqslant t < \infty$ the strict inequality $\sigma^*(t) > \sigma_e$ is satisfied.

Let us admit that the equality $\sigma^*(t_1) = \sigma_e$ is achieved for some finite value of the time t_1 . Since the function $\sigma^*(t)$ is monotonic, then $\sigma^*(t) \equiv \sigma_e$ for $t \geqslant t_1$. But then a $t_2 > t_1$ can be mentioned such that for $t \geqslant t_2$ in the unloading domain $\sigma(h, t) \equiv \sigma_e$, $\varepsilon(h, t) \equiv \varepsilon_e$, and $g(h, t) \equiv g_e$. Without difficulty we then reduce the momentum equation (2.1) for any $t \geqslant t_2$ to the condition $\omega(t) = 0$, which can be satisfied only in the trivial case $p_e = \sigma_m$.

If $\sigma_e = \sigma_s$, then by using the representation (1.7), we reduce (2.4) to the form

$$\omega(\alpha^*/g_0) = (\sigma_s - p_e)\,\alpha^*/g_0 + Q_1(t) + Q_2(t) + S_1(t) + S_2(t) + \Delta \qquad (2.6)$$

$$Q_1(t) = \frac{k_n}{2g_0}\int_{\alpha^*}^{\beta^*} \Lambda^n(\beta)\,d\beta, \quad Q_2(t) = \frac{1}{2g_0}\int_{\alpha^*}^{\beta^*} \theta_n(\beta)\,\Lambda^n(\beta)\,d\beta$$

Later, the cases $p_e > \sigma_s$, $p_e = \sigma_s$ and $p_e < \sigma_s$ are examined individually.

3. The case $p_e > \sigma_s$. Let us show first that the strict inequality $\sigma_e > \sigma_s$ holds in this case. To do this we turn to the evident relationship

$$[\sigma^*(t_M) - \sigma^*(t_N)] - \int_{\sigma^*(t_N)}^{\sigma^*(t_M)} \frac{g_0}{g}\,d\sigma = 2\,[p(t_A) - \sigma^*(t_N)] \qquad (3.1)$$

where t_N, t_M and t_A are ordinates of the intersections of the unloading line and the t axis with the straight lines $\alpha = \text{const}$ and $\beta = \text{const}$, whose segments AN and AM are shown in the sketch. The left side of (3.1) is not positive since $\sigma^*(t_M) \geqslant \sigma^*(t_N)$ and we have $g^*(t) < g_0$ for $\tau \leqslant t < \infty$. Since $p(t_A) \geqslant p_e$ and $p_e = \sigma_s + \delta$ $(\delta > 0)$, then the inequality $\sigma^*(t_N) \geqslant \sigma_s + \delta$ should be satisfied. We have $\sigma^*(t_N) \to \sigma_e$ as $t_N \to \infty$. Taking this last inequality into account, we arrive at the inequality $\sigma_e \geqslant \sigma_s + \delta$, and hence to the strict inequality $\sigma_e > \sigma_s$, QED.

Since $g_e \neq g_0$, then the asymptotic equality $\alpha^*(t) \sim (g_0 - g_e)\,t$, $t \to \infty$ can be written and then the estimate $R_2(t) = o(R_1(t))$, $t \to \infty$ can be obtained. Equation (2.5) can be reduced to the form

$$\omega(\alpha^*/g_0) = (\sigma_e - p_e)\,\alpha^*/g_0 + [1 + r_1(t)]\,R_1(t) + \Delta$$
$$r_1(t) \to 0 \quad \text{for} \quad t \to \infty$$

If the left and right sides of this equation are divided by t and t is then allowed to become infinite, we arrive at the important equality $\sigma_e = p_e$.

Let us insert the function

$$\lambda(\beta) = \beta\Lambda(\beta) \qquad (3.2)$$

into the expression defining $R_1(t)$, and apply the integral theorem of the mean. Moreover, taking into account the equality $\sigma_e = p_e$, we obtain

$$\omega\left(\frac{\alpha^*}{g_0}\right) = \frac{g_0 - g_e}{2g_0 g_e}\lambda(\zeta)[1 + r_1(t)]\ln\frac{\beta^*}{\alpha^*} + \Delta, \quad \alpha^* \leqslant \zeta \leqslant \beta^* \tag{3.3}$$

$$\Delta \to 0, \; r_1(t) \to 0 \quad \text{for} \quad t \to \infty$$

Let

$$\lim_{t\to\infty} \omega(t) = C, \quad C < \infty \tag{3.4}$$

Then it follows from (3.3) that

$$\lim_{\zeta\to\infty} \lambda(\zeta) = D, \quad D = \frac{2g_0 g_e C}{(g_0 - g_e)\ln[(g_0 + g_e)/(g_0 - g_e)]} \tag{3.5}$$

This result permits us to arrive at the conclusion that the unloading line for $p_e > \sigma_s$ and compliance with condition (3.4), has the oblique asymptote

$$h = g_e t - b, \quad b = g_e F(p_e) + 2l_2 g_e D \,/\, (g_0 + g_e) \tag{3.6}$$

The quantity C is the area of the domain bounded by the line segments $\sigma = p(t)$ and $\sigma = p_e$ and located to the right of the intersection of these lines (Fig. 1). If $C = \infty$, there is no asymptote.

On the basis of (1.4), (1.6), (3.2), the asymptotic equality $(g_0 - g_e)\beta^* \sim (g_0 + g_e)\alpha^*$, $t \to \infty$ and (3.5), we have

$$f_1(\alpha) \sim \frac{1}{2\rho_0 g_0}\int_0^{p_e}\frac{g_0 + g}{g}d\sigma + \frac{D}{2\rho_0}\left(\frac{1}{g_e} - \frac{1}{g_0}\right)\frac{1}{\alpha}, \quad \alpha \to \infty$$

$$f_2(\beta) \sim \frac{1}{2\rho_0 g_0}\int_0^{p_e}\frac{g_0 - g}{g}d\sigma + \frac{D}{2\rho_0}\left(\frac{1}{g_e} - \frac{1}{g_0}\right)\frac{1}{\beta}, \quad \beta \to \infty$$

According to the first formula in (1.2) and the formulas just obtained for $f_1(\alpha)$ and $f_2(\beta)$, the velocity of the end of the rod varies according to the asymptotic law

$$u(0, t) \sim -\frac{1}{\rho_0}\int_0^{p_e}\frac{d\sigma}{g} - \frac{D}{\rho_0 g_0}\left(\frac{1}{g_e} - \frac{1}{g_0}\right)\frac{1}{t}, \quad t \to \infty$$

Using (3.2), the asymptotic equality $g_e\beta^*(t) \sim (g_0 + g_e)\varphi(t)$, $t \to \infty$, and (3.5), we obtain

$$\sigma_*(h) \sim p_e + \frac{g_e D}{g_0 + g_e}\frac{1}{h}, \quad h \to \infty$$

It should be noted that the reduced formulas are valid even for $\sigma_s = 0$, i.e., when the curve mapping the loading law $\sigma(\varepsilon)$ has no initial linear section, where the slope of the parallel lines to the strain axis, by which unloading is realized, can also differ from the slope of the curve $\sigma(\varepsilon)$ at the origin to this same axis. If equality of the slopes mentioned is conserved, the asymptotic formulas obtained are valid only for $p_e > 0$; if such an equality is not conserved, and the former slope is greater than the latter, then the stress at the end of the rod can even decrease to zero. For instance, assuming $p_e = 0$ and letting g_0 tend to infinity, we arrive at the asymptotic of the solution of the unloading wave problem examined in detail in [5], according to whose assumption the relation between the stress and strain is nonlinear during the loading but the unloading of each element of the rod occurs under invariant strain.

4. The case $p_e = \sigma_s$. Firstly, it is easy to see by a proof by contradiction that $\sigma_e = \sigma_s$ in this case.

Furthermore, let us show that $\alpha^* (t) \to \infty$ as $t \to \infty$ in the case under consideration. To do this, we return to the figure and let the point N tend to infinity along the unloading line. If it is assumed that $\lim \alpha^* (t_N) < \infty$ as $t_N \to \infty$, then according to the equality $\alpha^* (t_N) = \beta^* (t_M)$, we will have $\lim t_M < \infty$ as $t_N \to \infty$. But then, the inequality $\lim p (t_A) < \sigma_s$, which contradicts the initial assumption $p_e = \sigma_s$, follows from (3.1). The second possibility $\alpha^* (t_N) \to \infty$ as $t_N \to \infty$ should therefore be realized.

This result permits arriving as the conclusion that the unloading line has no asymptote and permits writing the asymptotic equality

$$(g_0 - g^*) t \sim \alpha^*, \quad t \to \infty \tag{4.1}$$

and obtaining the estimate $Q_2 (t) = o (Q_1 (t)),\ t \to \infty$.

The asymptotic equality

$$nk_n g_0 t \Lambda^{n-1} (\beta^*) \sim \alpha^*, \quad t \to \infty \tag{4.2}$$

follows from the asymptotic equality (4.1) in an obvious manner, and can be used to obtain the estimate $S_2 (t) = o (S_1 (t)),\ t \to \infty$.

Taking account of the estimates obtained and also of the equality $p_e = \sigma_s$, we reduce (2.6) to the form

$$\omega (\alpha^* / g_0) = [1 + q_1 (t)] Q_1 (t) + [1 + s_1 (t)] S_1 (t) + \Delta \tag{4.3}$$
$$\Delta \to 0, \quad q_1 (t) \to 0, \quad s_1 (t) \to 0 \quad \text{for} \quad t \to \infty$$

Let us introduce the function

$$\Psi (\beta) = \beta \Lambda^n (\beta) \ln (\beta / L) \tag{4.4}$$

into the consideration, where L is a constant with the dimensionality of a length. If this function is used and the integral theorem of the mean is applied, then we can write

$$Q_1 (t) = \frac{k_n}{2g_0} \Psi (\zeta) \ln \mu (t), \quad \mu (t) = \frac{\ln (\beta^* / L)}{\ln (\alpha^* / L)}, \quad \alpha^* \leqslant \zeta \leqslant \beta^* \tag{4.5}$$

On the basis of (4.4), (4.2), the asymptotic equality $\beta^* \sim 2g_0 t,\ t \to \infty$, and the relationship resulting from (2.2) and (2.4)

$$S_1 (t) \sim \frac{(n-1) k_n}{2g_0} \frac{\Psi (\beta^*)}{\ln (\beta^* / L)}, \quad t \to \infty \tag{4.6}$$

we have the expression

$$\frac{1}{\mu (t)} = \frac{1}{n} + \frac{2g_0}{nk_n} S_1 (t) [1 + o(1)] \frac{\ln [\Psi (\beta^*) / \rho_0{}^n g_0^{2n} L]}{\Psi (\beta^*)} + o(1) \tag{4.7}$$
$$t \to \infty$$

which is useful for the investigation of the function $\mu(t)$ in (4.5).

The subsequent analysis will be performed under the condition (3.4). Let us proceed to determine the limit of the function $\Psi(\beta)$ as $\beta \to \infty$ under this condition, by assuming it to exist.

Let us assume that $\Psi(\beta) \to \infty$ as $\beta \to \infty$. Since condition (3.4) requires that the non-negative functions $Q_1(t)$ and $S_1(t)$ be bounded as $t \to \infty$, we then obtain $\mu(t) \to 1$ as $t \to \infty$ from (4.5), but another result from (4.7): $\mu(t) \to n$ as $t \to \infty$, which indicates the erroneousness of the assumption made relative to the behavior of the function $\Psi(\beta)$ at infinity.

Let $\Psi(\beta) \to 0$ as $\beta \to \infty$. Then it follows from (4.6), (4.3) and (4.5) that $S_1(t) \to 0$, $Q_1(t) \to C$, $\mu(t) \to \infty$ as $t \to \infty$. On the other hand, on the basis of the asymptotic equality (4.2) and the identity $(\alpha^* / \beta^*) \equiv (\beta^* / L)^{\nu(t)}$, $\nu(t) = \mu^{-1}(t) - 1$, the general formula

$$\Lambda^n(\beta^*) \sim \left(\frac{2}{nk_n}\right)^{n/(n-1)} \left(\frac{\beta^*}{L}\right)^{n\nu(t)/(n-1)}, \quad t \to \infty$$

can be established. Since the function $\mu(t)$ grows without limit as $t \to \infty$ according to the assumption made, then by using this formula, the estimate $[\Lambda(\beta^*) / \rho_0 g_0^2]^n < (\beta^* / L)^{-(1+\varkappa)}$ can be obtained for sufficiently large t, where $\varkappa$ is a positive constant satisfying the condition $\varkappa < 1 / (n-1)$. For sufficiently large β we therefore have the estimate $[\Lambda(\beta) / \rho_0 g_0^2]^n < (\beta / L)^{-(1+\varkappa)}$. However, if this estimate is valid, then we will have $Q_1(t) \to 0$ as $t \to \infty$ which contradicts the deduction obtained above the behavior of the function $Q_1(t)$ at infinity.

Thus, the limit of the function $\Psi(\beta)$ can only differ from zero by a finite quantity when $\beta \to \infty$. In the presence of this limit, the following passage to the limit are satisfied: $\lim S_1(t) = 0$, $\lim Q_1(t) = C$, and $\lim \mu(t) = n$ as $t \to \infty$. Using the relationship (4.5), we obtain

$$\lim_{\zeta \to \infty} \Psi(\zeta) = \frac{2g_0 C}{k_n \ln n}$$

A number of required asymptotic formulas

$$f_1(\alpha) \sim \frac{\sigma_s}{\rho_0 g_0} + \frac{C}{\rho_0 \ln n} \frac{1}{\alpha \ln(\alpha/L)}, \quad \alpha \to \infty$$

$$f_2(\beta) \sim \frac{C}{\rho_0 \ln n} \frac{1}{\beta \ln(\beta/L)}, \quad \beta \to \infty$$

$$\varphi(t) \sim g_0 t - nk_n g_0 \left(\frac{C}{k_n \ln n}\right)^{(n-1)/n} \left\{\frac{t}{[\ln(g_0 t/L)]^{n-1}}\right\}^{1/n}, \quad t \to \infty$$

$$u(0, t) \sim -\frac{\sigma_s}{\rho_0 g_0} - \frac{2C}{\rho_0 g_0 \ln n} \frac{1}{t \ln(g_0 t/L)}, \quad t \to \infty$$

$$\sigma_*(h) \sim \sigma_s + \left[\frac{g_0 C}{k_n \ln n} \frac{1}{h \ln(h/L)}\right]^{1/n}, \quad h \to \infty$$

can now be written without difficulty.

Setting $n = 2$ and $\sigma_s = 0$, we arrive at formulas known from [6].

5. The case $p_e < \sigma_s$. As in the preceding case $\sigma_e = \sigma_s$. However, in contrast to the preceding case, the unloading line always has the asymptote $h = g_0 t - b$. If the opposite is assumed, then $\alpha^*(t_N) \to \infty$ as $t_N \to \infty$ (Fig. 1),

and therefore, $t_A \to \infty$ and $t_M \to \infty$ also. But then we arrive at the following contradictory result: the left side of the equality (3.1) tends to zero as $t_N \to \infty$ and the right side to a limit different from zero. The existence of the asymptote for $p_e = 0$ was detected in [3], as is known. Attention was turned in [7] to the fact of the existence of an asymptote under the condition $p_e < \sigma_s$.

Since

$$g_0 - g^* \sim 2g_0 B / \beta^*, \quad B = b - g_0 F(\sigma_s), \quad t \to \infty$$

in the case under consideration, then by using (1.7) we obtain the asymptotic representation

$$f_2(\beta) \sim \left[\frac{k_n}{2\rho_0 g_0}\left(\frac{2B}{nk_n}\right)^{n/(n-1)}\right]\left(\frac{1}{\beta}\right)^{n/(n-1)}, \quad \beta \to \infty$$

The representation established in [4] for $n = 2$ and $p_e = 0$ hence results as a particular case.

The following asymptotic formulas

$$f_1(\alpha) - \frac{1}{\rho_0 g_0} p\left(\frac{\alpha}{g_0}\right) \sim \left[\frac{k_n}{2\rho_0 g_0}\left(\frac{2B}{nk_n}\right)^{n/(n-1)}\right]\left(\frac{1}{\alpha}\right)^{n/(n-1)}, \quad \alpha \to \infty$$

$$u(0, t) + \frac{1}{\rho_0 g_0} p(t) \sim -\left[\frac{k_n}{\rho_0 g_0}\left(\frac{2B}{nk_n g_0}\right)^{n/(n-1)}\right]\left(\frac{1}{t}\right)^{n/(n-1)}, \quad t \to \infty$$

$$\sigma_*(h) \sim \sigma_s + \left(\frac{B}{nk_n}\frac{1}{h}\right)^{1/(n-1)}, \quad h \to \infty$$

evidently also hold for any $p_e < \sigma_s$.

It is not possible to determine the quantity b as a result of an asymptotic investigation, however, an inequality useful in estimating the quantity b can be indicated. To do this, we pass to the limit in (2.4) as $t \to \infty$ but first setting $\sigma_e = \sigma_s$. Since we have $S_1(t) \to 0$, $S_2(t) \to 0$, $\alpha^*(t) \to b$, $\beta^*(t) \to \infty$ as $t \to \infty$, then in the limit we obtain

$$\omega\left(\frac{b}{g_0}\right) = (\sigma_s - p_e)\frac{b}{g_0} + \int_b^\infty f_2(\beta)\, d\beta - \int_{p_e}^{\sigma_s} F(\sigma)\, d\sigma$$

Taking into account that the function $f_2(\beta)$ takes on positive values for $g_0\tau \leqslant \beta < \infty$ and $b > g_0\tau$, we arrive at the desired inequality

$$\omega\left(\frac{b}{g_0}\right) + \int_{p_e}^{\sigma_s} F(\sigma)\, d\sigma > (\sigma_s - p_e)\frac{b}{g_0}$$

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